AN INEQUALITY RELATING TO TRIPLE PRODUCT INTEGRALS OF LAGUERRE FUNCTIONS

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ABSTRACT

If $L_n(x)$ is the nth Laguerre polynomial and $\lambda_n(x) = e^{-\frac{1}{2}x} L_n$ $(n = 0, 1, \ldots)$, then we can expand the functions $\{\lambda_n^2(x)\}$ over $(0, \infty)$ in terms of the set $\{\lambda_n(x)\}$, i.e., $\lambda_r^2(x) = \Sigma_{t=0}^{\infty} K_{rt}\lambda_r(x)$. In this paper we prove an old-standing conjecture that $(-1)^t K_{rt} > 0$ for $0 \leqq t \leqq r$ $(r = 0,1,\ldots)$; i.e., that, in the sense defined by Trench, the set $\{\lambda_n^2\}$ is alternating with respect to the set $\{\lambda_r(x)\}$.

1. Introduction

If we start from the Laguerre polynomials

$$L_n(x) = \sum_{\alpha=0}^{n} (-1)^{\alpha} \binom{n}{\alpha} \frac{x^n}{n!}$$

and define $\lambda_n(x) = e^{-\frac{1}{2}x}L_n(x)$, then it is well known that these form an orthonormal set, complete in $L^2(0, \infty)$. The question of linearizing a product of two such functions, that is,

$$\lambda_r(x)\lambda_s(x) = \sum_{t=0}^{\infty} C_{rst}\lambda_t(x) \tag{1.1}$$

led, in an earlier paper [3], to the study of the coefficients C_{rst}. These can be obtained in the usual way, as

$$C_{rst} = \int_0^{\infty} \lambda_r(x)\lambda_s(x)\lambda_t(x)\,dx. \tag{1.2}$$

It was conjectured [3] that $(-1)^t C_{rrt} > 0$ for $0 \leqq t \leqq r$. Subsequently [2] the coefficients C_{rrt} were evaluated asymptotically for large r in two special cases, t fixed and $t = r$, and the results were found to be in accord with this conjecture.

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In this paper we shall prove the conjecture by induction, using the recurrence relations established in [2].

More generally if $\{\phi_n\}$, $\{\psi_n\}$ are two sets of functions defined in some domain and $\{\psi_n\}$ is complete in some function space which includes the members of $\{\phi_n\}$ then we can always form the expansion

$$\phi_h \sim \sum_{k=0}^{\infty} A_{hk}\psi_k \qquad (h = 0, 1, \cdots).$$

Trench has introduced the concept of $\{\phi_n\}$ being *alternating* with respect to $\{\psi_n\}$ if $(-1)^k A_{hk} \geqq 0$ for $0 \leqq k \leqq h$. Hence what we shall prove in this paper is that the set $\{\lambda_n^2\}$ is alternating with respect to $\{\lambda_n\}$.

2. Recurrence relations

We define

$$K_{rt} = C_{rrt} = \int_0^\infty \lambda_r^2(x)\lambda_t(x)\,dx \tag{2.1}$$

$$S_{rt} = (-1)^t K_{rt} \tag{2.2}$$

$$E_r = K_{rr} \tag{2.3}$$

$$T_r = S_{rr} = (-1)^r E_r\,. \tag{2.4}$$

We note that while C_{rst} is clearly symmetric in r, s, t, the coefficients K_{rt}, S_{rt} are not, in general, symmetric.

The following recurrence relations have been established [2].

$$(X_{r,s,t}) \qquad \begin{aligned} &3C_{rst} - (C_{r-1,s,t} + C_{r,s-1,t} + C_{r,s,t-1}) \\ &\quad - (C_{r,s-1,t-1} + C_{r-1,s,t-1} + C_{r-1,s-1,t}) + 3C_{4-1,s-1\,t-1} = 0, \end{aligned}$$

unless $r = s = t = 0$, in which case

$$C_{0,0,0} = 2/3, \tag{2.5}$$

$$(Y_{r,t}) \qquad \begin{aligned} &r\{9K_{r,t} - 6K_{r,t-1} + K_{r,t-2}\} - (2r-1)\{5K_{r-1,t} - 6K_{r-1,t-1} + \\ &\quad + 5K_{r-1,t-2}\} + (r-1)\{K_{r-2,t} - 6K_{r-2,t-1} + 9K_{r-2,t-2}\} = 0 \end{aligned}$$

$$(Z_{r\cdot t}) \qquad \begin{aligned} &9tK_{r,t} - 3(2t-1)K_{r,t-1} + (t-1)K_{r,t-2} - tK_{r-1,t} \\ &\quad + 3(2t-1)K_{r-1,t-1} - 9(t-1)K_{r-2,t-2} = 0 \end{aligned}$$

and

$$(U_{r,t})\qquad 3t(t-1)K_{r,t}-2(t-1)(5t-8r-9)K_{r,t-1} + 2(t-2)(5t-8r-14)K_{r,t-3}-3(t-2)(t-3)K_{r,t-4}=0.$$

We shall also have to make frequent use of $(X_{n,n,n})$ which is easily seen to reduce to

$$(V_n)\qquad K_{n,n-1}+K_{n-1,n} = E_n+E_{n-1}.$$

The subsequent proof will be based mainly on the relations $(X)-(V)$. To simplify reference to these equations we have denoted them, as above, by $(X_{r,s,t})$, $(Y_{r,t})$, $(Z_{r,t})$, $(U_{r,t})$, (V_n) respectively. When using them for other values of the subscripts, we shall indicate the fact by appropriate changes in the subscripts of X, Y, Z, U, V.

Some care has to be taken when we use the above relations for small values of r, s, t where some of the subscripts can become negative. It turns out that these can be taken care of quite simply, with one exception, by simply setting equal to zero all terms $C_{\alpha,\beta,\gamma}$ or $K_{\alpha,\beta}$ im which any of the subscripts is negative. The one exception is when $r=s=t=0$ in (X); (note: in our notation $(X_{0,0,0})$). In this case (X) has to be replaced by (2.5).

3. The case $t = r = n$: proof that $T_n > 0$

We shall prove the following relations:

$$(3.1)\qquad 9(n+1)^2T_{n+1} = (9n^2+1)(T_n+T_{n-1})-9(n-1)^2T_{n-2}$$

$$(3.2)\qquad 9n^2T_n = 9(n-1)^2T_{n-2}+(T_0+9T_1+T_{n-1})+2\sum_{i=1}^{n-2}T_i$$

for $n = 3,4,5\cdots$. Since T_0, T_1, T_2 are known to be positive [3], [2], it follows by induction from (3.2) that $T_n > 0$ $(n = 0,1,2,3,\cdots)$. It is also easily seen that (3.2) follows by induction from (3.1), and so it only remains to prove (3.1).

Proof of (3.1). From $(X_{n-1,n,n})$, $(X_{n-1,n-1,n})$ respectively we obtain

$$(3.3)\quad 3K_{n,n-1}-(K_{n,n-2}+2K_{n-1,n})-(K_{n-1,n-1}+2C_{n,n-1,n-2})+3K_{n-1,n-2}=0$$

and

$$(3.4)\quad 3K_{n-1,n}-(2C_{n,n-1,n-2}+K_{n-1,n-1})-(2K_{n-1,n-2}+K_{n-2,n})+3K_{n-2,n-1}=0$$

Subtraction of (3.4) from (3.3) leads to

$$(3.5)\qquad K_{n,n-2}-K_{n-2,n} = 3K_{n,n-1}-5K_{n-1,n}+5K_{n-1,n-2}-3K_{n-2,n-1}$$

and, changing n to $(n+1)$ in (3.5), gives

$$K_{n+1,n-1} - K_{n-1,n+1} = 3K_{n+1,n} - 5K_{n,n+1} + 5K_{n,n-1} - 3K_{n-1,n}. \tag{3.6}$$

Consider now the fourteen relations $(Y_{n+1,n+1})$, $(Y_{n,n+1})$, $(Z_{n,n+2})$, $(Z_{n,n+1})$ $(Z_{n,n})$, $(Z_{n-1,n+1})$, $(Z_{n-1,n})$, $(U_{n,n+2})$, $(U_{n-1,n+2})$, (V_m), (V_{n-1}), (V_{n+1}), (3.5), and (3.6). We can see by inspection that they contain exactly thirteen *off-diagonal* terms K_{rt}, that is, terms in which $r \neq t$. These can be eliminated successively from the fourteen equations by elementary methods and we obtain finally

$$9(n+1)^2 E_{n+1} + (9n^2+1)(E_n - E_{n-1}) - 9(n-1)^2 E_{n-2} = 0. \tag{3.7}$$

Substituting from (2.4) in (3.7) immediately yields (3.1).

We have already noted that (3.2) follows from (3.1) by induction, and hence also that $T_n > 0$.

4. The case $t = 0$

It follows from (2.1) and (2.2) that

$$S_{ro} = K_{ro} = \int_0^\infty \lambda_r^2(x)dx > 0. \tag{4.1}$$

Moreover,

$$\begin{aligned} \sum_{r=0}^{\infty} K_{ro} u^r &= F(u,0) \\ &= 2(9-10u+u^2)^{-\frac{1}{2}} \\ &= \frac{2}{3}\sum_{r=0}^{\infty}\left(\frac{1}{3}u\right)^r P_r\left(\frac{5}{3}\right) \end{aligned} \tag{4.2}$$

where $P_r(\ )$ is the rth Legendre polynomial, and hence

$$S_{ro} = K_{ro} = \frac{2}{3^{r+1}} P_r\left(\frac{5}{3}\right). \tag{4.3}$$

Later we shall also need the fact that $S_{r,o} < S_{r-1,o}$ $(r = 1,2,\cdots)$. This follows immediately from (4.3). For if we write $p_r = P_r(5/3)$, than, by the standard recurrence relation for Legendre polynomials [1], we have

$$(r+1)p_{r+1} = \frac{5}{3}(2r+1)p_r - rp_{r-1}. \tag{4.4}$$

If now $p_r/p_{r-1} < 3$ then, by (4.4), we shall also have

$$\frac{p_{r+1}}{p_r} < \frac{5}{3}\cdot\frac{2r+1}{(r+1)} - \frac{r}{3(r+1)}$$

(4.5)

$$= \frac{9r+5}{3(r+1)} < 3.$$

Since $p_1/p_0 = 5/3 < 3$, it follows that $p_r/p_{r-1} < 3$ for all r and hence, by (4.3),

(4.6) $$S_{r,o} < S_{r-1,o}.$$

5. Proof of complete conjecture

In this section we shall prove that, for $r \geqq t > 0$,

$$S_{r,t} > 0.$$

Since we have seen in Section 4 that this also holds for $t = 0$ and, in Section 3 that $E_r = S_{rr} > 0$, this will complete the proof of our conjecture.

We shall begin by proving the following relations:

(5.1) $$9(t+1)(S_{t-1,t-1} - S_{t,t-1}) = 3(t-1)(S_{t-1,t-2} - S_{t,t-2}) + 8S_{t-1,t-1}$$

(5.2) $$9tS_{r,t} = 3(2t-1)(S_{r-1,t-1} - S_{r,t-1}) + (t-1)(S_{r-1,t-2} - S_{r,t-2}) + 8(t-1)S_{r-1,t-2} + tS_{r-1,t}$$

(5.3) $$9(2r-t+1)(S_{r-1,t-1} - S_{r,t-1}) = (2r+t-3)(S_{r-2,t-1} - S_{r-1,t-1}) + 8S_{r-1,t-1} + 3(t-1)(S_{r-2,t-2} - S_{r,t-2})$$

where $r, t = 0, 1, 2, \cdots$, with the usual proviso that terms in which either of the subscripts is negative have to be set equal to zero. We shall then deduce our final result from these three relations by induction.

PROOF OF (5.1), (5.2), and (5.3). If we write down $(Y_{r,t+1})$, (Y_{rt}), $(Z_{r-1,t+1})$, $(Z_{r-1,t})$ we see that the 4×4 determinant of the coefficients of $K_{r-2,t+1}$, $K_{r-2,t}$, $K_{r-2,t-1}$, $K_{r-2,t-2}$ in these equations vanishes, and so all four terms can be eliminated. This leads to

$(W_{r,t+1})$ $$9r(t+1)K_{r,t+1} - (r+4)(t+1)K_{r-1\,t+1} - 3r(11t-7)K_{r\,t} + 3(3rt-7r+4t+4) + 3(3rt-7r+4t+4)K_{r-1,t} + r(19t-17)K_{r,t-1} - (27rt-17r-4t+4)K_{r-1,t-1} - 3r(t-1)K_{r,t-2} + 3(9r-4)(t-1)K_{r-1\,t-2} = 0.$$

We adjoin to this the six relations $(Y_{r+1,t+1})$, $(Y_{r+1,t})$, $(Z_{r+1,t+1})$, $(Z_{r+1,t})$, $(Z_{r,t+1})$, $(Z_{r,t})$ and, from the combined set of seven, eliminate the five terms $K_{r+1,t+1}$, $K_{r+1,t-2}$, $K_{r-1,t-2}$, $K_{r,t+1}, K_{r-1,t+1}$. This yields the two equations

$$(5.4)\quad [9(r+1)K_{r+1,t}-(10r-4t+5)K_{r,t}-3(r+1)K_{r+1,t-1}]+rK_{r-1,t}+ + 3(2r+1)K_{r,t-1}-3rK_{r-1,t-1}-4(t-1)K_{r,t-2}=0 \quad \text{and}$$

$$(5.5)\quad 9[9(r+1)K_{r+1,t}-(10r-4t+5)K_{r,t}-3(r+1)K_{r+1,t-1}]+ + (9r-32t)K_{r-1,t}+2r-3\ 3(82t+41)K_{r,t-1}-3(73r-32t)K_{r-1,t-1}- - 4(t-1)K_{r,t-2}=0$$

and $9\times(5.4)$–(5.5) yields simply

$$(5.6)\quad tK_{r-1,t}-3(2r-t+1)K_{r,t-1}+3(2r-t)K_{r-1,t-1}-(t-1)K_{r,t-2}=0.$$

We now see that both $K_{r,t-2}$ and $K_{r-1,t}$ can be eliminated between (5.6) and $(Z_{r,t})$, giving

$$(5.7)\quad 3tK_{r,t}-(2r+t)K_{r,t-1}+(2r+t-1)K_{r-1,t-1}-3(t-1)K_{r-1,t-2}=0.$$

If we write $r+1$ for r in (5.6), multiply the resulting equation by 3, and subtract from (5.7), we obtain

$$(5.8)\quad 9(2r-t+3)K_{r+1,t-1}-2(10r-4t+9)K_{r,t-1}+(2r+t-1)K_{r-1,t-1}+ + 3(t-1)(K_{r+1,t-2}-K_{r-1,t-2})=0.$$

Now set $r=t-1$ in (5.8) and eliminate $K_{t-2,t-1}$ from the resulting relation by means of $(V_{t-1,t-1})$. We obtain

$$(5.9)\quad 9(t+1)(K_{t-1,t-1}-K_{t,t-1})+3(t-1)(K_{t-1,t-2}-K_{t,t-2})-8K_{t-1,t-1}=0$$

that is, by (2.2), the relation (5.1).

Also, substituting from (2.2) in $(Z_{r,t})$ immediately yields (5.2), while, if we replace by r by $r-1$ in (5.8) and again use (2.2), we obtain (5.3).

We shall now use (5.1)–(5.3) to prove our conjecture. This has already been proved for the special cases $t=0$, $t=r$ $(r=0,1,2,\cdots)$, and it remains to prove that $S_{rt}>0$ for $1\leqq t\leqq r-1$, $r\geqq 2$. The proof will be by induction. Indeed, we shall show that for r,t in this range,

$$(5.10)\quad S_{r-1,t}>S_{r,t}>0.$$

The proof will be in two stages:

i. (5.10) holds for $t=1$;

ii. if (5.10) holds for all $r < \rho$, $0 \leqq t \leqq r$, and for $r = \rho$, $0 \leqq t \leqq \tau < \rho - 1$, then it will also hold for $r = \rho$, $t = \tau + 1$.

PROOF OF (5.10i). Set $t = 1$ in (5.2), and we obtain

$$9S_{r,1} = 3(S_{r-1,0} - S_{r,0}) + S_{r-1,1} > S_{r-1,1} \tag{5.11}$$

by (4.6), and hence

$$S_{r,1} > 9^{-r+1} S_{1,1} > 0. \tag{5.12}$$

Also, if we substitute $(r + 1)$ for r in (5.3) and set $t = 2$, we obtain

$$\begin{aligned} 9(2r + 1)(S_{r,1} - S_{r+1,1}) &= (2r + 1)(S_{r-1,1} - S_{r,1}) + \\ &\quad + 8S_{r,1} + 3(S_{r-1,0} - S_{r+1,0}). \end{aligned} \tag{5.13}$$

But it follows from (4.5) that

$$S_{r-1,0} > S_{r,0} > S_{r+1,0} \tag{5.14}$$

and hence, by (5.12), (5.14), (5.13) we obtain

$$\begin{aligned} S_{r,1} - S_{r+1,1} &> \frac{1}{9}(S_{r-1,1} - S_{r,1}) \\ &> \frac{1}{9^{r-1}}(S_{1,1} - S_{2,1}). \end{aligned} \tag{5.15}$$

Now it is easily verified, for example, by expanding the low order terms of (2.6), that

$$S_{1,1} = \frac{2}{27}, \; S_{2,1} = \frac{10}{243}$$

and so by (5.15),

$$S_{r,1} - S_{r+1,1} > 8.3^{-2r-3} > 0 \tag{5.16}$$

for $r \geqq 1$.

PROOF OF (5.10ii). Set $r = \rho$, $t = \tau + 1$ in (5.2), and it follows at once, by the inductive hypothesis, that $S_{\rho,\tau+1} > 0$.

To prove that $S_{\rho,\tau+1} < S_{\rho-1,\tau+1}$ we have to distinguish two cases.

Case 1. $t < \rho - 2$. Set $r = \rho$, $t = \tau + 2$ in (5.3) and we obtain

$$\begin{aligned} &9(2\rho - \tau - 1)(S_{\rho-1,\tau+1} - S_{\rho,\tau+1}) \\ &= (2\rho + \tau + 1)(S_{\rho-2,\tau+1} - S_{\rho-1,\tau+1}) + 8S_{\rho-1,\tau+1} + 3(\tau + 1)(S_{\rho-2,\tau} - S_{\rho,\tau}) \end{aligned} \tag{5.17}$$

and each term on the right-hand side of (5.17) is positive by the inductive hypothesis.

Case 2. $\tau = \rho - 2$, that is, we have to prove that $S_{\rho,\rho-1} < S_{\rho-1,\rho-1}$, but this follows at once from (5.3) if we set $t = \rho$.

REFERENCES

1. A. Erdélyi, *Higher Transcendental Functions II*, McGraw-Hill, New York, 1953.

2. J. Gillis, *Integrals of products of Laguerre polynomials*, SIAM J. Math. Anal. (in press).

3. J. Gillis, and M. Shimshoni, *Triple product integrals of Laguerre functions*, Math. Comp **16** (1962), 50.

4. W. F. Trench, *Nonnegative and alternating expansion of one set of orthogonal polynomials in terms of another*, SIAM J. Math. Anal. 4 (1973), 111.

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